Abstract

This addendum collects an operational atlas of transfer packages for Presentation Theory. It is not presented as a sequence of closed new theorems, but as a reusable map of contexts, descriptions, observables, verification data, fibres, scales, and overhead functions. Each network indicates which objects should be compared, which observables should be pulled back or pushed forward, which verification data must remain controllable, and which fibre geometry measures the loss of information. The purpose is to provide a complete repertoire of templates from which later work can extract transfer theorems with explicit hypotheses and budgets.

1 Status and Reading Guide

The atlas should be read as a foundational support document. An arrow between two theories does not automatically assert the existence of an equivalence or of a complete transfer theorem. It records instead the type of compiler, observational pullback, verification datum, fibre, and overhead that would have to be controlled in order to obtain a theorem. In this sense the central object is not an informal translation between domains, but the controlled package of access to information.

For each network one should distinguish three levels. The first is descriptive: it identifies contexts that are already naturally connected in mathematical practice. The second is formal: it specifies which maps between descriptions, observables, verification data, and fibres should enter the package. The third is demonstrative: it requires effective estimates for the overheads and suitable stability conditions. Only the third level produces an autonomous theorem; the first two provide a grammar for formulating it without losing the budgets.

The terminology used here follows the current version of the theory: observables are the functions, invariants, or procedures that measure an object through a declared channel; observable profiles are the finite or partial images produced by those channels; verification data are the controllable pieces of information that justify a property in the source or in the target; fibres are classes of objects indistinguishable with respect to a fixed profile. Overhead functions measure how descriptive size, observational cost, verification cost, and fibre-navigation cost grow under transfer.

2 Common Grammar of the Atlas

2.1 Presentational Contexts

A node of the atlas is not merely a theory (T), but a theory equipped with an access structure:

$$\mathfrak{P} = (\mathcal{X}, \Gamma, \mathfrak{O}, \Sigma, \mathfrak{V}, \mathfrak{F}, \mathfrak{M}).$$

Where:

$$\mathcal{X}$$

is the class of objects.

$$\Gamma = (\mathcal{D}, \rho, \kappa)$$

is the presentation system: descriptions, realization, and cost.

$$\mathfrak{O}$$

is the system of observables.

$$\Sigma$$

is the operational structure: sums, products, compositions, limits, gluing, mutations, evolutions.

$$\mathfrak{V}$$

is the system of verification data.

$$\mathfrak{F}$$

is the geometry of fibres.

$$\mathfrak{M}$$

is the possible structure of measure, density, probability, or distribution on the objects.

The complexity of an object is not an absolute property. It is relative to:

$$\text{how the object is described}$$

$$\text{how the object is observed}$$

$$\text{how the object is verified}$$

$$\text{how one moves among equivalent descriptions.}$$

2.2 Transfer package

A transfer package from a context $\mathfrak{P}_A$ to a context $\mathfrak{P}_B$ is denoted by:

$$\mathfrak{T} : A \longrightarrow B.$$

The ideal form contains:

$$(F, \widehat{F}, F^*, \Sigma, V, \Phi, \mu).$$

Where:

$$F : \mathcal{X}_A \to \mathcal{X}_B$$

transports objects.

$$\widehat{F} : \mathcal{D}_A \to \mathcal{D}_B$$

compiles descriptions.

$$F^* : \mathfrak{O}_B \to \mathfrak{O}_A$$

pulls back observables.

$$\Sigma$$

controls the behavior with respect to operations.

$$V$$

transports verification data.

$$\Phi$$

controls fibres, moves, and bridge profiles.

$$\mu$$

transports measures, densities, distributions, or probabilistic limits.
The transfer package carries an overhead vector:

$$\mathbf{f}_{\mathfrak{T}} = (f_{\mathrm{pres}}, f_{\mathrm{obs}}, f_{\mathrm{op}}, f_{\mathrm{ver}}, f_{\mathrm{fib}}, f_{\mathrm{meas}}, f_{\mathrm{lim}}, f_{\mathrm{reg}}, f_{\mathrm{scale}}).$$

There is not a single cost. There are many costs.

2.3 Basic Resources

We use the following resources:

$$P = \text{presentational cost}$$

$$O = \text{observational cost}$$

$$V = \text{verification cost}$$

$$B = \text{cost of navigation in the fibre}$$

$$M = \text{metric, probabilistic, or measure-theoretic cost}$$

$$R = \text{regularity}$$

$$S = \text{scale}$$

$$\varepsilon = \text{precision}$$

$$T = \text{observation time}$$

$$\Lambda = \text{frequency or spectral cutoff}$$

$$d = \text{degree}$$

$$q = \text{quantifier rank}$$

$$n = \text{dimension, size, number of samples, or number of states.}$$

2.4 General Principles

Principle 1: one transfers access, not the object

$$\boxed{\text{a map transports objects; a transfer package transports access to the objects.}}$$

An ordinary reduction says:

$$x \mapsto F(x).$$

A transfer package says:

$$\text{descriptions of } x \mapsto \text{descriptions of } F(x),$$

$$\text{observables on } F(x) \mapsto \text{observables on } x,$$

$$\text{verification data in the target} \mapsto \text{verification data in the source},$$

$$\text{fibre of the target} \leftrightarrow \text{fibre of the source}.$$

Principle 2: every bounded observation induces fibres

Given an observable, or a family of bounded observables:

$$\mathcal{O}_{\leq r},$$

one obtains a partition:

$$x \sim_r y \quad \Longleftrightarrow \quad O(x) = O(y) \text{ for every } O \in \mathcal{O}_{\leq r}.$$

The hidden complexity lives in the classes:

$$[x]_r.$$

Therefore:

$$\boxed{\text{the geometry of the fibre is the geometry of forgotten information.}}$$

Principle 3: transfer packages compose

If:

$$A \xrightarrow{\mathfrak{T}} B \xrightarrow{\mathfrak{S}} C,$$

then:

$$A \xrightarrow{\mathfrak{S}\circ\mathfrak{T}} C.$$

The overheads compose:

$$f_{\mathfrak{S}\circ\mathfrak{T}} \leq f_{\mathfrak{S}} \circ f_{\mathfrak{T}}.$$

This produces **transfer graphs**.

Principle 4: Morita theory with budgets

Two theories may be considered equivalent, not absolutely, but up to a declared class of overhead functions $\mathcal{H}$:

$$A \simeq_{\mathcal{H}} B.$$

This means that there are packages in both directions:

$$A \to B, \qquad B \to A,$$

whose compositions are equivalent to the identity with controlled overhead.
This is a **quantitative Morita theory**.

Principle 5: every theory has many observable profiles

Every theory produces many observable profiles:

spectra

moments

entropies

cohomologies

finite quotients

truncations

samples

invariants

categories

operators

kernel

tensors.

The task of the atlas is to measure the cost of passing from one observable profile to another.

3 The Main Transversal Hubs

The atlas is large, but many edges pass through a small number of universal hubs.

3.1 Hub of Observables

$$\mathcal{O}(\mathcal{X})$$

is the algebra, category, or system of observables of a theory.
Every object gives an evaluation:

$$x \mapsto \mathrm{ev}_x : \mathcal{O}(\mathcal{X}) \to A.$$

Every map:

$$F : \mathcal{X} \to \mathcal{Y}$$

induces a pullback:

$$F^* : \mathcal{O}(\mathcal{Y}) \to \mathcal{O}(\mathcal{X}).$$

This is the contravariant core of the transfer theory.

3.2 Hub of Fibres

Every observational map:

$$F : \mathcal{X} \to \mathcal{Y}$$

produces fibres:

$$F^{-1}(y).$$

Fibres should be studied with:

moves

bridge profiles

diameters

mixing

singularities

moduli

internal probabilities

navigation algorithms.

Universal examples:

functions with the same low Fourier coefficients

domains with the same first eigenvalues

$$\text{metrics with the same truncated heat trace}$$

$$\text{PDEs with the same input-output data}$$

$$\text{neural networks representing the same function}$$

$$\text{distributions with same moments}$$

$$\text{graphs with the same small subgraphs}$$

$$\text{strutture with same theory logic up to a quantifier rank } q.$$

3.3 Hub of Operators

Many theories are transported into operators:

$$X \mapsto T_X.$$

Examples:

$$M \mapsto \Delta_M$$

$$G \mapsto L_G$$

$$T : X \to X \mapsto U_T$$

$$\text{Markov process} \mapsto L$$

$$\text{kernel } K(x, y) \mapsto T_K$$

$$\text{PDE} \mapsto P(x, D).$$

Then:

$$T \mapsto \mathrm{Spec}(T),$$

$$T \mapsto e^{tT},$$

$$T \mapsto (z - T)^{-1},$$

$$T \mapsto \mathrm{Tr}(e^{-tT}).$$

3.4 Hub of Kernels

$$K(x,y)$$

appears in:

$$\text{PDE}$$

$$\text{machine learning}$$

$$\text{operator theory}$$

$$\text{probability}$$

$$\text{graph limits}$$

$$\text{quantum mechanics}$$

$$\text{integral geometry.}$$

Scheme:

$$\text{object} \to \text{kernel} \to \text{operator} \to \text{spectrum}$$
$$\to \text{features.}$$

3.5 Spectral Hub

$$\text{Spec}$$

connects:

$$\text{graphs}$$

$$\text{varieties}$$

$$\text{operators}$$

$$\text{dynamics}$$

$$\text{quantum mechanics}$$

$$\text{automorphic forms}$$

$$\text{zeta functions}$$

$$\text{trace formulas.}$$

Scheme:

$$\text{object} \to \text{operator} \to \text{spectrum} \to \text{zeta/trace}$$
$$\to \text{arithmetic or geometry.}$$

3.6 Hub of Moments

$$m_\alpha = \int x^\alpha, d\mu.$$

Moments connect:

$$\text{probability}$$

$$\text{statistics}$$

$$\text{SDP}$$

$$\text{operator algebras}$$

$$\text{quantum correlations}$$

$$\text{inverse problems.}$$

Scheme:

$$\text{object} \to \text{moments} \to \text{semidefinite relaxation} \to \text{verification datum.}$$

3.7 Entropic Hub

$$H, \quad S, \quad \mathrm{Ent}, \quad h_{\mathrm{top}}, \quad h_\mu.$$

Connects:

$$\text{probability}$$

$$\text{dynamics}$$

$$\text{compression}$$

$$\text{statistical mechanics}$$

$$\text{optimal transport}$$

$$\text{learning theory}$$

$$\text{quantum information.}$$

Scheme:

$$\text{object} \to \text{distribution} \to \text{entropy} \to \text{concentration/compression/mixing.}$$

3.8 Sheaf Hub

$$\mathcal{F}.$$

Connects:

$$\text{topology}$$

$$\text{PDE microlocali}$$

$$\text{database}$$

$$\text{CSP}$$

$$\text{distributed systems}$$

$$\text{algebraic geometry}$$

$$\text{persistence.}$$

Scheme:

$$\text{local data} \to \text{sheaf} \to \text{cohomology} \to \text{obstruction.}$$

3.9 Tensorial Hub

$$T \in V_1 \otimes \cdots \otimes V_k.$$

Connects:

$$\text{algebra}$$

$$\text{quantum information}$$

$$\text{PDE kernels}$$

$$\text{statistics}$$

$$\text{machine learning}$$

$$\text{complexity theory}$$

$$\text{cohomology.}$$

Scheme:

$$\text{struttura multilineare} \to \text{tensore} \to \text{rank} \to \text{lower bound.}$$

3.10 Hub of Convex Verification Data

Includes:

SOS

SDP

dual verification data

Lyapunov functions

entropy verification data

barrier verification data

moment relaxations.

Scheme:

$$\text{truth} \to \text{verification datum} \to \text{verifiable bound}.$$

3.11 Hub of Approximations

All of these are variants of the same scheme:

$$X \mapsto X_{\leq r}.$$

Examples:

Taylor jets

Fourier cutoffs

finite quotients

mesh refinements

moment truncations

logical quantifier-rank truncations

spectral truncations

prefixes of words

finite samples.

3.12 Hub of Dualities

Dualities are bidirectional transfer packages.

Examples:

$$\text{compact spaces} \leftrightarrow \text{commutative } C^*\text{-algebras}$$

$$\text{locally compact abelian groups} \leftrightarrow \text{Pontryagin duals}$$

$$\text{corpi convessi} \leftrightarrow \text{support functions}$$

$$\text{measures} \leftrightarrow \text{moments/characteristic functions}$$

$$\text{D-modules} \leftrightarrow \text{perverse sheaves}$$

$$\text{symplectic} \leftrightarrow \text{mirror algebraic geometry}$$

$$\text{syntax} \leftrightarrow \text{semantics}$$

$$\text{automorfo} \leftrightarrow \text{Galois.}$$

The presentational question is not only whether the duality exists, but:

how much do descriptions, observables, and verification data grow across the duality?

4 Atlas of Networks

We now pass to the networks themselves.

Each network is a graph of theories. Each edge is a possible transfer package.

5 A. Foundational, Logical, and Syntactic Networks

5.1 A1. Network syntax–semantics–CSP–database–proof complexity

Nodes:

equational theories

universal algebras

Lawvere theories

operadi

monadi

$$\text{term rewriting}$$

$$\text{CSP}$$

$$\text{database theory}$$

$$\text{finite moof the theory}$$

$$\text{proof complexity.}$$

Main edges:

$$\text{presentation equazionale} \to \text{category of models}$$

$$\text{equational theory} \to \text{CSP template}$$

$$\text{database schema} \leftrightarrow \text{finite-limit theory}$$

$$\text{term rewriting} \to \text{proof complexity.}$$

Resources:

$$P = \text{number of symbols, arity, term length}$$

$$O = \text{quantifier rank, width, treewidth}$$

$$V = \text{derivation length, degree, width}$$

$$B = \text{distance between equivalent presentations.}$$

Central idea:

$$\text{query of database} \Longleftrightarrow \text{observable}$$

$$\text{proof derivation} \Longleftrightarrow \text{bridge in the fibra of the presentations equivalent.}$$

5.2 A2. Network types–proofs–categories–homotopy–formalization

Nodes:

$$\lambda\text{-calculus}$$

$$\text{type theory}$$

$$\text{dependent type theory}$$

$$\text{proof assistants}$$

$$\text{categories}$$

$$\text{toposes}$$

$$\text{homotopy type theory}$$

$$\infty\text{-categories}$$

$$\text{rewriting}$$

$$\text{normalization}$$

$$\text{proof data.}$$

Edges:

$$\text{proof} \to \text{term}$$

$$\text{type theory} \to \text{category}$$

$$\text{category} \to \text{internal language}$$

$$\text{homotopy type} \to \text{space/simplicial object}$$

$$\text{formal proof} \to \text{checkable proof data.}$$

Resources:

$$P = \text{context dimension, universe levels, costruttori}$$

$$O = \text{depth of types, truncation level}$$

$$V = \text{type-checking time, proof-term length}$$

$$B = \text{distance between equivalent proofs, normalization length.}$$

Central question:

$$\text{can a mathematically simple result be formally expensive?}$$

5.3 A3. Network continuous logic–functional analysis–probability

Nodes:

$$\text{continuousus logic}$$

$$\text{metric structures}$$

$$\text{Banach spaces}$$

$$C^*\text{-algebras}$$

$$\text{probability algebras}$$

$$\text{tracial von Neumann algebras}$$

$$\text{ultraproducts metrici}$$

$$\text{stability theory}$$

$$\text{definability.}$$

Edges:

$$X \mapsto \mathrm{Th}_{\leq q}(X)$$

$$(X_n) \mapsto \prod_{\mathcal{U}} X_n$$

$$A \mapsto \text{tracial moments}$$

$$\text{Banach space} \mapsto \text{metric theory.}$$

Resources:

$$O = \text{quantifier rank, modulus of continuity, precision}$$

$$V = \text{elementary approximation verification data}$$

$$B = \text{fibres of indistinguishable structures up to a rank } q.$$

5.4 A4. Network o-minimality–PDE tame–real geometry–algorithms

Nodes:

$$\text{o-minimal structures}$$

$$\text{semialgebraic sets}$$

$$\text{subanalytic sets}$$

$$\text{Pfaffian functions}$$

$$\text{definable manifolds}$$

$$\text{tame topology}$$

$$\text{real analytic geometry}$$

$$\text{optimization}$$

$$\text{differential equations definable.}$$

Edges:

$$f \mapsto \mathrm{Graph}(f)$$

$$\text{definable set} \mapsto \text{Betti numbers, cell decomposition}$$

$$\text{ODE definibile} \mapsto \text{flow definibile}$$

$$\text{optimization problem} \mapsto \text{critical point data.}$$

Resources:

$$P = \text{format definibile, degree, chain length}$$

$$O = \text{cell dimension, Betti complexity}$$

$$V = \text{stratification verification data, monotonicity}$$

$$B = \text{fibres with the same tame decomposition.}$$

Idea:

$$\text{analysis with controlled topological complexity.}$$

5.5 A5. Network finite precision–computable analysis–real complexity

Nodes:

$$\text{computable real numbers}$$

$$\text{represented spaces}$$

$$\text{Type-2 computability}$$

$$\text{computable analysis}$$

$$\text{Weihrauch degrees}$$

$$\text{numerical algorithms}$$

$$\text{real RAM models}$$

$$\text{interval computation}$$

$$\text{constructive analysis.}$$

Edges:

$$x \in \mathbb{R} \mapsto \text{name } (q_n)$$

$$F : X \to Y \mapsto \text{realizer}$$

$$\text{analytic theorem} \mapsto \text{Weihrauch degree}$$

$$\text{numerical approximation} \mapsto \text{interval verification data.}$$

Resources:

$$O = 2^{-n} \text{ precision}$$

$$V = \text{modulus of continuity, convergenza, enclosure}$$

$$B = \text{fibre of different names of the same object.}$$

5.6 A6. Network PDE–logic–proof mining–constructive analysis

Nodes:

proof mining

constructive analysis

functional interpretation

fixed point theorems

ergodic theorems

PDE existence proofs

compactness arguments

rates of convergence

metastability.

Edges:

$$\text{classical proof} \mapsto \text{effective bound}$$

$$\text{convergence theorem} \mapsto \text{metastability bound}$$

$$\text{compactness proof} \mapsto \text{modulus extraction}$$

$$\text{PDE existence} \mapsto \text{approximation scheme} + \text{bound}.$$

Idea:

$$\text{qualitative theorem} \to \text{quantitative profile} \to \text{algorithm} \to \text{verification datum}.$$

6 B. Algebraic, Representational, and Homological Networks

6.1 B1. Network representations–moduli–invariants–deformations

Nodes:

finitely presented groups

associative algebras

Lie algebras

quiver

rings

tensor categories

representation varieties

character varieties

GIT quotients

moduli stacks

deformation theory.

Edges:

$$G \to \mathrm{Rep}_n(G)$$

$$A \to \mathrm{Mod}_n(A)$$

$$Q \to \mathrm{Rep}(Q, \mathbf{d})$$

$$\mathcal{C} \to K_0(\mathcal{C})$$

$$X \to \mathfrak{g}_X.$$

Observables:

traces of words

characters

semi-invariants

stabilizers

orbit closures

deformation cohomology.

Fibres:

non-isomorphic objects with the same invariants

$$\text{orbits with the same closure}$$

$$\text{moduli with the same decategorification.}$$

6.2 B2. Universal Tensorial Network

Nodes:

$$\text{finite-dimensional algebras}$$

$$\text{matrix multiplication}$$

$$\text{groups and group algebras}$$

$$\text{cohomology and cup product}$$

$$\text{multilinear forms}$$

$$\text{probability discrete}$$

$$\text{graphical models}$$

$$\text{quantum states}$$

$$\text{tensor networks}$$

$$\text{PDE kernels}$$

$$\text{complexity of bilinear maps.}$$

Edges:

$$A \mapsto T_A$$

where T_A is the multiplication tensor.

$$X \mapsto T_{H^*(X)}$$

cup product in cohomology.

$$p(x_1, \ldots, x_n) \mapsto T_p$$

distribution come tensore.

$$\psi \mapsto T_\psi$$

stato quantistico multipartito.

$$K(x,y) \mapsto T_K$$

kernel discretizzato or approssimato.
Observables:

$$\text{rank}$$

$$\text{border rank}$$

$$\text{slice rank}$$

$$\text{partition rank}$$

$$\text{flattening rank}$$

$$\text{bond dimension}$$

$$\text{entanglement entropy.}$$

Idea:

every composizione bilineare or multilineare genera a transfer tensoriale.

6.3 B3. Network linear algebra–matroids–codes–data structures–rank methods

Nodes:

$$\text{matrices}$$

$$\text{matroids}$$

$$\text{linear codes}$$

$$\text{linear data structures}$$

$$\text{communication matrices}$$

$$\text{incidence geometries}$$

$$\text{polynomial method}$$

$$\text{linear circuits}$$

$$\text{proof complexity algebrica.}$$

Edges:

$$\text{problem} \mapsto A_P$$

matrice of incidenza, comunicazione or vincoli.

$$A \mapsto M(A)$$

matroid.

$$A \mapsto C_A$$

linear code.

$$\text{data structure} \mapsto A = RM$$

fattorizzazione sparsa.

$$\text{polynomial proof} \mapsto \text{linear map of coefficients.}$$

Resources:

$$P = \text{dimension, sparsity, field}$$

$$O = \text{rank, local rank, support size}$$

$$V = \text{dependence verification data, dual codewords}$$

$$B = \text{distance between representations of the same matroid.}$$

6.4 B4. Network deformations–cohomology–obstructions–derived moduli

Nodes:

algebraic varieties

schemes

complexes

dg-algebras

dg-Lie algebras

L_∞-algebras

Hochschild cohomology

André–Quillen cohomology

$$\text{deformation functors}$$

$$\text{derived stacks}$$

$$\text{obstruction theory.}$$

Edges:

$$X \mapsto \mathfrak{g}_X$$

$$\mathfrak{g} \mapsto MC(\mathfrak{g})$$

$$MC(\mathfrak{g}) \mapsto H^*(\mathfrak{g})$$

$$\text{moduli problem} \mapsto \text{derived moduli stack.}$$

Resources:

$$P = \text{dimension of the complesso, differentials}$$

$$O = \text{observed cohomological degree}$$

$$V = \text{nilpotent order of the verified deformation}$$

$$B = \text{gauge equivalence, homotopies, quasi-isomorphisms.}$$

Idea:

$$\text{nonlinear classification} \to \text{linear data plus higher obstructions.}$$

6.5 B5. Network homological algebra–functional analysis–derived categories

Nodes:

$$\text{complexes}$$

$$\text{derived categories}$$

$$\text{triangulated categories}$$

$$\text{dg-categories}$$

$$\text{Ext/Tor}$$

$$\text{Hochschild homology}$$

cyclic homology

topological cyclic homology

operator K-theory

index theory.

Edges:

$$A \mapsto D(A)$$

$$M, N \mapsto \mathrm{Ext}^*(M, N)$$

$$A \mapsto HH_*(A)$$

$$A \mapsto HC_*(A)$$

$$\text{smooth algebra} \mapsto \text{de Rham-type invariants.}$$

Fibres:

$$\text{categories with same invariants omologici troncati.}$$

7 C. Geometric, Topological, and Categorical Networks

7.1 C1. Network tropical–matroidal–polyhedral–optimization

Nodes:

algebraic varieties

ideali

schemes

Newton polytopes

tropical varieties

matroids

oriented matroids

fans

$$\text{cluster algebras}$$

$$\text{linear programming}$$

$$\text{integer programming}$$

$$\text{phylogenetic geometry.}$$

Edges:

$$I \mapsto \mathrm{Trop}(I)$$

$$I \mapsto \mathrm{in}_w(I)$$

$$\text{linear arrangement} \mapsto \text{matroid}$$

$$\text{varieties positiva} \mapsto \text{polytope}$$

$$\text{cluster algebra} \mapsto \text{fan combinatorics.}$$

Resources:

$$P = \text{degree, number of monomials, bit-size}$$

$$O = \text{dimension of the fan, number of cones, matroidal rank}$$

$$V = \text{tropical verification data, Groebner bases}$$

$$B = \text{distance of mutazione, distance in the Gröbner fan.}$$

7.2 C2. Network sheaf–cosheaf–local consistency–persistence–distributed systems

Nodes:

$$\text{sheaves}$$

$$\text{cosheaves}$$

$$\text{CSP locali}$$

$$\text{database joins}$$

$$\text{bundle theory}$$

$$\text{Cech cohomology}$$

$$\text{persistent homology}$$

$$\text{distributed computing}$$

$$\text{sensor networks}$$

$$\text{local-to-global proof systems}$$

$$\text{descent theory.}$$

Edges:

$$\text{CSP instance} \mapsto \mathcal{F}_{\text{solutions}}$$

$$\text{database} \mapsto \text{sheaf of compatible records}$$

$$\text{cover} \mapsto \check{C}^{\bullet}(\mathcal{U}, \mathcal{F})$$

$$\text{filtered data} \mapsto \text{persistence module}$$

$$\text{distributed task} \mapsto \text{simplicial carrier/sheaf.}$$

Idea:

$$\text{CSP width} \leftrightarrow \text{cohomological obstruction degree}$$
$$\leftrightarrow \text{database join complexity} \leftrightarrow \text{distributed task impossibility.}$$

7.3 C3. Network topology–groups–knots–TQFT–skein–mapping class groups

Nodes:

$$\text{complexes simpliciali}$$

$$\text{varieties PL}$$

$$\text{handle decompositions}$$

$$\text{knots and link}$$

$$\text{fundamental groups}$$

$$\text{quandles}$$

$$\text{skein modules}$$

$$\text{TQFT}$$

$$\text{mapping class groups}$$

$$\text{modular tensor categories}$$

$$\text{character varieties.}$$

Edges:

$$M \mapsto \pi_1(M)$$

$$M \mapsto C_\bullet(M)$$

$$K \mapsto \pi_1(S^3 \setminus K)$$

$$K \mapsto Q(K)$$

$$M \mapsto Z_{\mathcal{C}}(M)$$

$$K \mapsto \text{skein element.}$$

Observables:

$$\text{homology}$$

$$\text{gruppo fondamentale}$$

$$\text{colorazioni quandle}$$

$$\text{quantum invariants up to level } r.$$

7.4 C4. Network persistent topology–functional analysis–signals–PDE

Nodes:

$$\text{filtered spaces}$$

$$\text{persistent homology}$$

$$\text{persistence modules}$$

$$\text{barcodes}$$

$$\text{sheaf persistence}$$

$$\text{signals}$$

$$\text{images}$$

$$\text{solutions of PDE}$$

$$\text{random fields}$$

$$\text{multiparameter persistence.}$$

Edges:

$$f : X \to \mathbb{R} \mapsto X_{\le t_t}$$

$$X_t \mapsto H_k(X_t)$$

$$\text{module} \mapsto \text{barcode}$$

$$u(x,t) \mapsto \text{persistence over level/time}$$

$$\text{random field} \mapsto \text{random barcode.}$$

Central fibre:

$$f \mapsto \mathrm{Barcode}(f)$$

forgets a large part of the geometry.

7.5 C5. Network symplectic geometry–Floer–mirror symmetry–elliptic PDE

Nodes:

$$\text{symplectic manifolds}$$

$$\text{Hamiltonian dynamics}$$

$$\text{Lagrangian submanifolds}$$

$$\text{pseudo-holomorphic curves}$$

$$\text{Floer homology}$$

$$\text{Fukaya categories}$$

mirror symmetry

derived categories of coherent sheaves

Gromov–Witten theory

quantum cohomology.

Edges:

$$(M, \omega) \mapsto \text{Hamiltonian flow}$$

$$L_0, L_1 \mapsto CF^*(L_0, L_1)$$

$$J \mapsto \text{moduli of } J\text{-holomorphic curves}$$

$$X \mapsto QH^*(X)$$

$$\text{symplectic side} \mapsto \text{algebraic mirror.}$$

7.6 C6. Network Teichmüller–quasiconformal–complex dynamics

Nodes:

Riemann surfaces

Teichmüller spaces

moduli of curves

quasiconformal maps

mapping class groups

measured foliations

geodesic flows

complex dynamics

flat surfaces

interval exchange transformations.

Edges:

$$X \mapsto \mathcal{T}(X)$$

$$q \mapsto \text{horizontal/vertical foliations}$$

$$\text{flat surface} \mapsto \text{translation flow}$$

$$\text{mapping class} \mapsto \text{action on curves/homology}$$

$$\text{rational map} \mapsto \text{Julia set / lamination.}$$

7.7 C7. Network measure geometry–currents–varifold–fractals

Nodes:

measures of Hausdorff

rectifiability

currents

varifolds

minimal surfaces

free boundary

fractals

multifractal analysis

geometric flows

GMT compactness.

Edges:

$$S \mapsto \mathcal{H}^d \llcorner S$$

$$S \mapsto [S]$$

$$S \mapsto V_S$$

$$T \mapsto \partial T$$

$$\mu \mapsto \dim_H(\mu), \tau(q), f(\alpha).$$

Observables:

$$\text{scale}$$

$$\text{massa}$$

$$\text{flat norm}$$

$$\text{dimension}$$

$$\text{multifractal profile.}$$

7.8 C8. Network fractals–iterated dynamics–spectra–probability

Nodes:

$$\text{iterated function systems}$$

$$\text{self-similar sets}$$

$$\text{self-affine sets}$$

$$\text{fractal measures}$$

$$\text{analysis on fractals}$$

$$\text{spectral decimation}$$

$$\text{random fractals}$$

$$\text{multifractal formalism}$$

$$\text{substitution systems.}$$

Edges:

$$S_i \mapsto K$$

$$K \mapsto \mu_K$$

$$K \mapsto \Delta_K$$

$$\Delta_K \mapsto \mathrm{Spec}(\Delta_K)$$

$$\text{substitution} \mapsto \text{tiling dynamical system.}$$

8 D. Analytic Networks

This is the large analytic expansion of the atlas.

8.1 D1. Network distributions–Fourier–Sobolev–microlocal

Nodes:

smooth functions

L^p

distributions

measures

trasformata of Fourier

Sobolev spaces

Besov spaces

wavelets

operators pseudodifferenziali

wavefront sets

microlocal sheaves.

Edges:

$$u \mapsto \widehat{u}$$

$$u \mapsto |u|_{H^s}$$

$$u \mapsto WF(u)$$

$$P(x, D) \mapsto \sigma(P)$$

$$u \mapsto \langle u, \psi_{j,k} \rangle.$$

Fibres:

$$u \mapsto \widehat{u}|_{|\xi| \leq \Lambda}$$

forgets high frequencies.

$$u \mapsto WF(u)$$

forgets amplitudes and fine phases.

$$P \mapsto \sigma_{\text{prin}}(P)$$

forgets subprincipal symbols.

8.2 D2. Network elliptic PDE–regularity–geometry–topology

Nodes:

operators ellittici

problemi al bordo

Sobolev spaces

Fredholm theory

index

cohomology

K-theory

Riemannian geometry

spectral theory

invariants topologici.

Edges:

$$P \mapsto \sigma(P)$$

$$P \mapsto \text{ind}(P)$$

$$Pu = f \mapsto u = P^{-1}f$$

$$(M, E, P) \mapsto [\sigma(P)] \in K(T^*M)$$

$$P \mapsto \ker P, \text{coker}\, P.$$

Idea:

PDE ellittica $\to$ topology $\to$ non-invertibility or existence verification data.

8.3 D3. Network parabolic PDE–semigroups–probability–geometry

Nodes:

$$\text{heat equations}$$

$$\text{contractive semigroups}$$

$$\text{operators dissipativi}$$

$$\text{processi of Markov}$$

$$\text{Dirichlet forms}$$

$$\text{heat kernels}$$

$$\text{curvature-dimension}$$

$$\text{gradient flows}$$

$$\text{optimal transport.}$$

Edges:

$$L \mapsto e^{tL}$$

$$L \mapsto X_t$$

$$L \mapsto p_t(x, y)$$

$$p_t(x, y) \mapsto \mathrm{Tr}(e^{tL})$$

$$\partial_t u = Lu \mapsto \text{gradient flow}$$

$$\mathcal{E}(u, u) \mapsto \text{Dirichlet form} \mapsto \text{Markov process.}$$

8.4 D4. Network hyperbolic PDE–propagation–causality–scattering

Nodes:

$$\text{equazioni of the onde}$$

$$\text{operators iperbolici}$$

$$\text{geometry lorentziana}$$

$$\text{propagatori}$$

$$\text{fronti of onda}$$

$$\text{bicharacteristics}$$

$$\text{scattering}$$

$$\text{control theory}$$

$$\text{inverse boundary problems.}$$

Edges:

$$P \mapsto \mathrm{Char}(P)$$

$$\mathrm{Char}(P) \mapsto \text{flusso hamiltoniano}$$

$$u_0, u_1 \mapsto u(t)$$

$$u(t)|_{\partial M} \mapsto \text{boundary observation}$$

$$(M, g) \mapsto \Lambda_g.$$

Questions:

$$\text{how much time is needed to see a singularity?}$$

$$\text{which angular aperture is needed to reconstruct a geometry?}$$

8.5 D5. Network dispersive–nonlinear–scattering–solitons

Nodes:

$$\text{Schrödinger nonlinear}$$

$$\text{wave maps}$$

$$\text{Klein–Gordon}$$

$$\text{KdV}$$

$$\text{Strichartz estimates}$$

$$\text{profile decompositions}$$

$$\text{concentration compactness}$$

$$\text{scattering states}$$

$$\text{solitons.}$$

Edges:

$$u_0 \mapsto u(t)$$

$$u(t) \mapsto u_\pm$$

$$u_n \mapsto \phi_j, \lambda_j, x_j, t_j$$

$$u \mapsto \text{mass, energy, momentum}$$

$$\text{nonlinear PDE} \mapsto \text{renormalized effective dynamics.}$$

Fibres:

$$u_0 \mapsto (M(u_0), E(u_0))$$

has dynamically rich fibres.

8.6 D6. Network harmonic analysis–singular operators–additive combinatorics

Nodes:

$$\text{Fourier analysis}$$

$$\text{Calderón–Zygmund operators}$$

$$\text{maximal functions}$$

$$\text{restriction theory}$$

$$\text{decoupling}$$

$$\text{time-frequency analysis}$$

$$\text{additive combinatorics}$$

$$\text{incidence geometry}$$

$$\text{ergodic averages}$$

$$\text{PDE dispersive.}$$

Edges:

$$f \mapsto \widehat{f}|_{\Sigma}$$

$$f \mapsto Tf$$

$$\text{curvature} \mapsto \text{decay/dispersion estimates}$$

$$\text{decoupling} \mapsto \text{Vinogradov-type counting}$$

$$\text{incidence bound} \mapsto \text{operator norm bound.}$$

Scheme:

$$\text{curvature} \to \text{oscillation} \to \text{cancellation} \to \text{estimate} \\ \to \text{counting.}$$

8.7 D7. Network microlocal–symplectic–sheaf-theoretic–PDE

Nodes:

distributions

wavefront sets

Lagrangiane coniche

Fourier integral operators

symplectic geometry

microlocal sheaves

Fukaya categories

PDE propagation

contact geometry.

Edges:

$$u \mapsto WF(u)$$

$$P \mapsto H_{\sigma(P)}$$

$$\text{FIO} \mapsto \text{canonical relation}$$

$$\mathcal{F} \mapsto SS(\mathcal{F})$$

$$\text{Lagrangian} \mapsto \text{sheaf quantization.}$$

Idea:

$$\text{PDE singularities} \to \text{Legendrian geometry} \to \text{sheaf categories} \to \text{topological invariants.}$$

8.8 D8. Network spectral theory–scattering–zeta–traces

Nodes:

$$\text{operators autoaggiunti}$$

$$\text{non-self-adjoint operators}$$

$$\text{spectra discreti}$$

$$\text{spectra continui}$$

$$\text{resonances}$$

$$\text{scattering matrices}$$

$$\text{trace formulas}$$

$$\text{zeta functions}$$

$$\text{quantum chaos}$$

$$\text{inverse spectral geometry.}$$

Edges:

$$P \mapsto \mathrm{Spec}(P)$$

$$P \mapsto \zeta_P(s)$$

$$P \mapsto \mathrm{Tr}(e^{-tP})$$

$$\text{geodesic flow} \mapsto \text{trace formula}$$

$$V(x) \mapsto S_V(\lambda).$$

Fibres:

$$P \mapsto \lambda_1, \ldots, \lambda_N$$

has very large fibres at low energy.

8.9 D9. Network Riemannian geometry–Ricci flow–PDE geometric–topology

Nodes:

metric riemanniane

curvature tensors

Ricci flow

mean curvature flow

Yamabe flow

geometric analysis

singularity models

topology of the varieties

limit metric spaces.

Edges:

$$g \mapsto \mathrm{Rm}(g), \mathrm{Ric}(g), R(g)$$

$$g_0 \mapsto g(t)$$

$$g(t) \mapsto \text{singularity model}$$

$$(M, g) \mapsto \text{volume growth, heat kernel, spectrum}$$

$$(M, g_i) \mapsto \text{Gromov–Hausdorff limit.}$$

Scheme:

$$\text{geometry} \to \text{parabolic PDE} \to \text{monotonicity} \to \text{topology.}$$

8.10 D10. Network metric geometry–synthetic Ricci–optimal transport

Nodes:

metric measure spaces

Wasserstein geometry

curvature-dimension conditions

optimal transport

gradient flows

entropy

heat flow

concentration

Markov chains.

Edges:

$$(X, d, \mu) \mapsto \mathcal{P}_2(X)$$

$$\mu_0, \mu_1 \mapsto \pi^*$$

$$\mathrm{Ent}_\mu \mapsto \text{gradient flow}$$

$$(X, d, \mu) \mapsto \mathrm{CD}(K, N) \text{ profile}$$

$$\text{Markov chain} \mapsto \text{discrete transport geometry.}$$

Scheme:

$$\text{curvature} \leftrightarrow \text{convexity of the entropy} \leftrightarrow \text{heat contraction} \leftrightarrow \text{concentration} \leftrightarrow \text{mixing.}$$

8.11 D11. Network calculus of variations–Γ-convergence–materials–PDE

Nodes:

functionals

minimizzatori

Γ-convergence

relaxation

quasiconvexity

Young measures

microstructures

elasticity

free boundary problems

optimal design.

Edges:

$$F_\varepsilon \mapsto \Gamma\text{-}\lim F_\varepsilon$$

$$u_\varepsilon \mapsto \nu_x$$

$$\text{nonconvex energy} \mapsto \text{relaxed energy}$$

$$\text{material microstructure} \mapsto \text{effective tensor}$$

$$\text{shape} \mapsto \text{shape derivative.}$$

Fundamental fibre:

$$\text{microstructure} \mapsto \text{effective material}$$

has very large fibres.

8.12 D12. Network homogenization–multiscale–random media

Nodes:

PDEs with oscillating coefficients

mezzi periodici

mezzi casuali

H-convergence

G-convergence

two-scale convergence

correctors

effective equations

percolation

random conductance models.

Edges:

$$a(x/\varepsilon) \mapsto a_{\text{hom}}$$

$$u_\varepsilon \mapsto u_0$$

$$u_\varepsilon \mapsto (u_0, u_1(x, y))$$

$$\omega \mapsto a_{\text{hom}}(\omega)$$

random medium $\mapsto$ large-scale heat kernel.

Scheme:

micro $\to$ meso $\to$ macro.

8.13 D13. Network fluids–turbulence–vorticity–statistics

Nodes:

$$\text{Euler}$$

$$\text{Navier–Stokes}$$

$$\text{vorticity}$$

$$\text{boundary layers}$$

$$\text{turbulenza}$$

$$\text{Reynolds averaging}$$

$$\text{large eddy simulation}$$

$$\text{Onsager regularity}$$

$$\text{statistical solutions}$$

$$\text{kinetic theory.}$$

Edges:

$$u \mapsto \omega = \nabla \times u$$

$$u^{\nu} \mapsto u^{0}$$

$$u \mapsto \overline{u}$$

$$u \mapsto E(k)$$

$$\text{particle system} \mapsto \text{kinetic equation} \mapsto \text{fluid equation.}$$

Fibres:

$$u \mapsto E(k)$$

forgets phases and coherent vortices.

$$u \mapsto \overline{u}$$

forgets small scales.

8.14 D14. Network kinetic theory–Boltzmann–entropy–hydrodynamic limits

Nodes:

systems of particelle

Liouville equation

Boltzmann equation

Landau equation

Vlasov equation

Fokker–Planck

entropy methods

hydrodynamic limits

fluctuation theory.

Edges:

$$x_i, v_{i\,i=1}^{N} \mapsto f_N$$

$$f_N \mapsto f$$

$$f \mapsto \rho, u, T$$

$$\text{Boltzmann} \mapsto \text{Euler/Navier–Stokes}$$

$$f \mapsto H(f).$$

8.15 D15. Network continuous probability–Malliavin–SPDE–regularity

Nodes:

Brownian motion

martingales

diffusions

Malliavin calculus

Dirichlet forms

SPDE

Gaussian fields

random distributions

stochastic quantization.

Edges:

$$b, \sigma \mapsto X_t$$

$$X_t \mapsto L$$

$$L \mapsto \mathcal{E}$$

$$X \mapsto DX$$

noise $\mapsto$ random distribution $\mapsto$ SPDE solution.

8.16 D16. Network rough paths–regularity structures–renormalization

Nodes:

rough paths

controlled paths

branched rough paths

regularity structures

paracontrolled distributions

renormalization group

Hopf algebras

SPDE singulari

stochastic analysis.

Edges:

$$X \mapsto \mathbf{X} = (X, \mathbb{X}, \dots)$$

$$\mathbf{X} \mapsto \text{solution map}$$

$$\text{SPDE} \mapsto \text{regularity structure}$$

$$\text{diagramma divergente} \mapsto \text{counterterm}$$

$$\text{trees} \mapsto \text{Hopf algebra.}$$

Fundamental fibre:

$$\mathbf{X} \mapsto X$$

forgets iterated areas.
Idea:

$$\text{segnale grezzo} \not\Rightarrow \text{dynamics ben posta}$$

ma

$$\text{segnale arricchito} \Rightarrow \text{dynamics continua.}$$

8.17 D17. Network ergodic theory–transfer operators–thermodynamic formalism

Nodes:

$$\text{systems dinamici misurabili}$$

$$\text{systems topologici}$$

$$\text{subshifts}$$

$$\text{operators of Koopman}$$

$$\text{operators of Perron–Frobenius}$$

$$\text{entropy}$$

$$\text{pressione}$$

$$\text{measures Gibbs}$$

$$\text{zeta dinamiche}$$

$$\text{large deviations.}$$

Edges:

$$T \mapsto U_T : f \mapsto f \circ T$$

$$T \mapsto \mathcal{L}_\phi$$

$$T \mapsto h_\mu(T), h_{\text{top}}(T)$$

$$T \mapsto \zeta_T(z)$$

$$\text{partition} \mapsto \text{symbolic coding.}$$

8.18 D18. Network chaos–KAM–small divisors–localization

Nodes:

systems hamiltoniani

tori invariants

KAM theory

small divisors

Diophantine approximation

quasi-periodic Schrödinger operators

Anderson localization

Aubry–Mather theory

renormalization.

Edges:

$$H(I,\theta) \mapsto \omega(I)$$

$$\omega \mapsto \text{Diophantine exponent}$$

$$\text{perturbation} \mapsto \text{surviving tori}$$

$$\text{quasi-periodic potential} \mapsto \text{spectrum}$$

$$\text{cocycle} \mapsto \text{Lyapunov exponent.}$$

Idea:

$$\text{arithmetic fine} \to \text{stability dynamics} \to \text{spectral analysis.}$$

8.19 D19. Network integrable systems–Riemann–Hilbert–isomonodromy

Nodes:

equazioni integrabili

Lax pairs

spectral curves

inverse scattering

Riemann–Hilbert problems

isomonodromic deformations

Painlevé equations

tau functions

Frobenius manifolds

random matrix theory.

Edges:

$$\partial_t L = [P, L] \mapsto \mathrm{Spec}(L)$$

$$u(x) \mapsto \text{scattering data}$$

$$\text{scattering data} \mapsto \text{Riemann–Hilbert problem}$$

$$\text{monodromy data} \mapsto \text{isomonodromic flow}$$

$$\text{RHP} \mapsto \tau.$$

8.20 D20. Network complex analysis–SCV–pluripotential–PDE

Nodes:

holomorphic functions

pseudoconvex domains

$\bar{\partial}$-problem

Bergman spaces

Hardy spaces

plurisubharmonic functions

Monge–Ampère complex

Stein manifolds

complex dynamics

Kähler geometry.

Edges:

$$\Omega \mapsto K_{\Omega}(z, w)$$

$$\Omega \mapsto \bar{\partial}\text{-Neumann operator}$$

$$\varphi \mapsto (dd^c\varphi)^n$$

$$X \mapsto H^{p,q}_{\bar{\partial}}(X)$$

$$f : \mathbb{C}^n \to \mathbb{C}^n \mapsto \text{Green current.}$$

8.21 D21. Network potential–capacity–electrical networks–probability

Nodes:

potential classico

capacity

Dirichlet energy

electrical networks

random walks

Brownian motion

harmonic measure

Laplacian growth

potential theory on graphs.

Edges:

$$D \mapsto \mathrm{cap}(D)$$

$$f \mapsto \int |\nabla f|^2$$

$$G \mapsto \text{electrical network}$$

$$\text{network} \mapsto \text{effective resistance metric}$$

$$\text{domain} \mapsto \omega_{\partial D}^{x}.$$

8.22 D22. Network non-self-adjoint operators–pseudospectra–numerical stability

Nodes:

operators non normali

pseudospectra

semigroups

linear stability

$$\text{transient growth}$$

$$\text{fluid stability}$$

$$\text{numerical analysis}$$

$$\text{control.}$$

Edges:

$$A \mapsto \mathrm{Spec}(A)$$

$$A \mapsto |(z - A)^{-1}|$$

$$A \mapsto \mathrm{Pseudospec}_{\varepsilon}(A)$$

$$A \mapsto e^{tA}$$

$$A \mapsto \sup_t |e^{tA}|.$$

Principle:

$$\text{same spectrum} \not\Rightarrow \text{same dynamics.}$$

8.23 D23. Network approximation–wavelet–compressed sensing–numerical learning

Nodes:

$$\text{approximation theory}$$

$$\text{polynomial approximation}$$

$$\text{splines}$$

$$\text{wavelets}$$

$$\text{sparse approximation}$$

$$\text{compressed sensing}$$

$$\text{reduced basis methods}$$

$$\text{neural operators}$$

$$\text{numerical PDE}$$

$$\text{information-based complexity.}$$

Edges:

$$f \mapsto \text{best } n\text{-term approximation}$$

$$f \mapsto \langle f, \psi_\lambda \rangle$$

$$u_\mu \mapsto \text{reduced basis space}$$

$$Au = f \mapsto A_h u_h = f_h$$

$$\text{operator} \mapsto \text{learned operator surrogate.}$$

Scheme:

$$\text{regularity} \to \text{compressibility} \to \text{sampling} \to \text{algorithms}$$
$$\to \text{error verification data.}$$

8.24 D24. Network numerical analysis–FEM–mesh–verification data

Nodes:

$$\text{finite elements}$$

$$\text{finite volumes}$$

$$\text{spectral methods}$$

$$\text{mesh refinement}$$

$$\text{adaptive algorithms}$$

$$\text{a posteriori estimates}$$

$$\text{verified numerics}$$

$$\text{interval arithmetic}$$

$$\text{computer-assisted proofs.}$$

Edges:

$$\text{PDE} \mapsto \text{weak formulation}$$

$$\text{weak formulation} \mapsto \text{discrete variational problem}$$

$$u_h \mapsto \eta_h$$

$$\eta_h \mapsto \text{mesh refinement}$$

$$\text{floating computation} \mapsto \text{interval verification data.}$$

Scheme:

$$\text{analysis} \to \text{algorithm} \to \text{verification datum} \to \text{formalization.}$$

8.25 D25. Network control–observability–identification–optimization

Nodes:

$$\text{controlled dynamical systems}$$

$$\text{ODE}$$

$$\text{PDE controllate}$$

$$\text{Hamilton–Jacobi–Bellman}$$

$$\text{Pontryagin maximum principle}$$

$$\text{reachability}$$

$$\text{observability}$$

$$\text{system identification}$$

$$\text{mean field games}$$

$$\text{reinforcement learning.}$$

Edges:

$$\dot{x} = f(x, u) \mapsto \mathcal{R}_T(x_0)$$

$$\text{control problem} \mapsto \text{value function}$$

$$\text{value function} \mapsto \text{HJB equation}$$

$$\text{trajectory data} \mapsto \hat{f}$$

$$\text{multi-agent system} \mapsto \text{mean field limit}.$$

Fibre:

$$f \mapsto \text{input-output behavior}$$

may have very large fibres.

8.26 D26. Network functional analysis–Banach–operator ideals–asymptotic geometry

Nodes:

$$\text{Banach spaces}$$

$$\text{Hilbert spaces}$$

$$\text{operator ideals}$$

$$\text{type/cotype}$$

$$\text{local theory}$$

$$\text{basis constants}$$

$$\text{compact operators}$$

$$\text{nuclear operators}$$

$$\text{tensor norms}$$

$$\text{nonlinear embeddings}.$$

Edges:

$$X \mapsto E \subset X : \dim E \leq n$$

$$T : X \to Y \mapsto (s_n(T))$$

$$X \mapsto \text{type/cotype profile}$$

$$X, Y \mapsto X \widehat{\otimes}_\pi Y,\ X \widehat{\otimes}_\varepsilon Y$$

$$X \mapsto \mathrm{Lip}_0(X).$$

8.27 D27. Network interpolation–scales of spaces–adaptive regularity

Nodes:

$$L^p$$

Sobolev

Besov

Triebel–Lizorkin

Hardy

BMO

Lorentz

Orlicz

interpolation spaces

regularity estimates.

Edges:

$$(X_0, X_1) \mapsto [X_0, X_1]_\theta$$

$$(X_0, X_1) \mapsto (X_0, X_1)_{\theta,q}$$

$$f \mapsto \text{regularity profile } s \mapsto |f|_{X_s}$$

$$T : X_0 \to Y_0,\ T : X_1 \to Y_1 \mapsto T : X_\theta \to Y_\theta.$$

8.28 D28. Network analytic nonlocality–fractional operators–jump processes

Nodes:

fractional Laplacians

nonlocal PDE

integro-differential operators

stable processes

$$\text{nonlocal minimal surfaces}$$

$$\text{peridynamics}$$

$$\text{anomalous diffusion.}$$

Edges:

$$(-\Delta)^s \mapsto \text{stable Lévy process}$$

$$u \mapsto \int \frac{u(x)-u(y)}{|x-y|^{n+2s}}, dy$$

$$s \to 1$$

$$s \to 0$$

$$K(x,y) \mapsto \text{jump process.}$$

8.29 D29. Network memory–delay equations–history operators–systems

Nodes:

$$\text{delay differential equations}$$

$$\text{Volterra equations}$$

$$\text{fractional time derivatives}$$

$$\text{memory kernels}$$

$$\text{viscoelasticity}$$

$$\text{control with memory}$$

$$\text{non-Markovian processes}$$

$$\text{state augmentation.}$$

Edges:

$$K(t) \mapsto \int_0^t K(t-s)u(s), ds$$

$$\text{memory system} \mapsto \text{augmented Markovian system}$$

$$K \mapsto \widehat{K}(\lambda)$$

$$\text{fractional derivative} \mapsto \text{subordinated semigroup}.$$

Idea:

$$\text{the presentation of a system may include the amount of past needed.}$$

8.30 D30. Network algebraic analysis–D-modules–PDE–perverse sheaves

Nodes:

D-modules

linear PDE systems

holonomic modules

characteristic varieties

Riemann–Hilbert correspondence

perverse sheaves

monodromy

Fourier–Laplace transform

irregular singularities

Stokes data.

Edges:

$$\text{PDE system} \mapsto D/D\langle P_i \rangle$$

$$M \mapsto \mathrm{Char}(M)$$

$$M_{\mathrm{hol}} \mapsto \mathrm{Sol}(M)$$

$$\mathrm{Sol}(M) \mapsto \text{perverse sheaf}$$

$$\text{irregular connection} \mapsto \text{Stokes data}.$$

Scheme:

$$\text{linear PDEs} \leftrightarrow \text{noncommutative algebra} \leftrightarrow \text{constructible topology}.$$

8.31 D31. Network transseries–asymptotics–resurgence–ODE/PDE

Nodes:

$$\text{asymptotic expansions}$$

$$\text{transseries}$$

$$\text{resurgence}$$

$$\text{Borel summation}$$

$$\text{Stokes phenomena}$$

$$\text{singular ODE}$$

$$\text{WKB analysis}$$

$$\text{quantum curves}$$

$$\text{enumerative geometry.}$$

Edges:

$$f(\varepsilon) \mapsto \sum a_n \varepsilon^n$$

$$\sum a_n \varepsilon^n \mapsto \mathcal{B}f$$

$$\text{Borel singularities} \mapsto \text{Stokes constants}$$

$$\text{semiclassical ODE} \mapsto \text{WKB transseries}$$

$$\text{enumerative generating function} \mapsto \text{resurgent structure.}$$

Idea:

$$\text{funzione} \to \text{serie divergente} \to \text{Borel singularities} \to \text{Stokes data} \to \text{monodromy.}$$

8.32 D32. Network analytic QFT–renormalization–operator algebras–probability

Nodes:

Euclidean QFT

constructive field theory

renormalization group

Gaussian free fields

CFT

OPE algebras

stochastic quantization

AQFT

factorization algebras

perturbative QFT.

Edges:

$$S \mapsto e^{-S}\mathcal{D}\phi$$

$$\text{field} \mapsto \text{correlation functions}$$

$$\text{correlators} \mapsto \text{OPE coefficients}$$

$$\text{cutoff theory} \mapsto \text{renormalized theory}$$

$$\text{local net} \mapsto \text{operator algebra}.$$

Idea:

QFT has many presentations: Lagrangian, Hamiltonian, operator-algebraic, correlational, and categorical.

8.33 D33. Network causality–Lorentzian geometry–hyperbolic PDE–relativity

Nodes:

Lorentzian manifolds

causal structures

Einstein equations

wave equations

null geodesics

Penrose diagrams

conformal geometry

inverse problems

black hole scattering.

Edges:

$g \mapsto$ light cones

$g \mapsto \Box_g$

$\Box_g \mapsto$ propagator

null geodesic flow $\mapsto$ lens relation

Einstein equations $\mapsto$ constraint equations.

8.34 D34. Network condensed mathematics–derived functional analysis

Nodes:

topological vector spaces

locally convex spaces

Fréchet spaces

nuclear spaces

$$\text{Schwartz distributions}$$

$$\text{condensed mathematics}$$

$$\text{solid modules}$$

$$\text{derived functional analysis}$$

$$\text{analytic geometry over rings.}$$

Edges:

$$V \mapsto \underline{V}$$

$$\text{functional analytic problem} \mapsto \text{abelian categorical problem}$$

$$\text{distribution space} \mapsto \text{sheaf/condensed module}$$

$$\text{topological tensor product} \mapsto \text{derived tensor.}$$

Idea:

$$\text{absorbing functional analysis into a stable categorical language.}$$

9 E. Probability, Statistics, Information, Learning, and Applications

9.1 E1. Network probability–statistics–information–algebraic geometry

Nodes:

$$\text{distributions discrete}$$

$$\text{measures continue}$$

$$\text{moments}$$

$$\text{exponential families}$$

$$\text{graphical models}$$

$$\text{algebraic statistics}$$

$$\text{toric varieties}$$

information geometry

optimal transport

learning theory

coding theory.

Edges:

$$p \mapsto (m_\alpha(p))_{|\alpha|\le d}$$

graphical model $\mapsto$ conditional independence ideal

Bayesian network $\mapsto$ semialgebraic model

$$\mu, \nu \mapsto W_p(\mu, \nu)$$

code $\mapsto$ probability channel.

Central fibre:

$$\theta \mapsto p_\theta$$

measure identifiability and non-identifiability.

9.2 E2. Network information–entropy–compression–analysis

Nodes:

Shannon entropy

Rényi entropy

Fisher information

log-Sobolev inequalities

transport inequalities

compression

statistical estimation

large deviations

thermodynamic formalism

quantum information.

Edges:

$$\mu \mapsto H(\mu)$$

$$\mu \mapsto I(\mu)$$

$$\text{Markov semigroup} \mapsto \text{entropy dissipation}$$

$$\text{source} \mapsto \text{code length profile}$$

$$\rho \mapsto S(\rho).$$

Scheme:

$$\text{entropy} \leftrightarrow \text{heat} \leftrightarrow \text{optimal transport} \leftrightarrow \text{concentration} \leftrightarrow \text{compression} \leftrightarrow \text{learning}.$$

9.3 E3. Network large deviations–variational principles–limit PDE

Nodes:

probability measures

large deviation principles

rate functions

Hamilton–Jacobi equations

statistical mechanics

mean field limits

optimal control

rare events.

Edges:

$$X_\varepsilon \mapsto I$$

$$I \mapsto \text{variational problem}$$

$$\text{Markov process} \mapsto \text{Hamiltonian}$$

$$\text{rare event} \mapsto \text{optimal path}.$$

9.4 E4. Network statistical mechanics–Gibbs measures–phase transitions–complexity

Nodes:

spin systems

Gibbs measures

partition functions

cluster expansions

phase transitions

random fields

percolation

Ising/Potts models

statistical algorithms

complexity of counting.

Edges:

$$H(\sigma) \mapsto Z(\beta)$$

$$H \mapsto \mu_\beta$$

$$\mu_\beta \mapsto \text{correlation functions}$$

$$\text{spin model} \mapsto \text{graphical expansion}$$

$$\text{Gibbs sampler} \mapsto \text{Markov chain.}$$

9.5 E5. Network mathematical machine learning–kernels–operators–geometry

Nodes:

$$\text{kernel methods}$$

$$\text{RKHS}$$

$$\text{Gaussian processes}$$

$$\text{neural networks}$$

$$\text{neural tangent kernels}$$

$$\text{mean field limits}$$

$$\text{optimal transport}$$

$$\text{statistical learning theory}$$

$$\text{PAC-Bayes}$$

$$\text{information bottleneck.}$$

Edges:

$$k(x, y) \mapsto \mathcal{H}_k$$

$$\text{network} \mapsto \text{function class}$$

$$\text{wide network} \mapsto \text{kernel/mean-field limit}$$

$$\text{data distribution} \mapsto \text{risk functional}$$

$$\text{training dynamics} \mapsto \text{gradient flow.}$$

Central fibre:

$$\text{parameters} \mapsto \text{function}$$

has very large fibres.

9.6 E6. Network scientific machine learning–PDE discovery–operator learning

Nodes:

$$\text{PDE data}$$

$$\text{symbolic regression}$$

$$\text{operator learning}$$

$$\text{neural operators}$$

$$\text{physics-informed neural networks}$$

$$\text{reduced order models}$$

$$\text{Bayesian calibration}$$

$$\text{uncertainty quantification}$$

$$\text{digital twins.}$$

Edges:

$$\text{data } u(x,t) \mapsto \hat{P}$$

$$P \mapsto \mathcal{S}_P$$

$$\mathcal{S}_P \mapsto \widehat{\mathcal{S}}$$

$$\text{surrogate} \mapsto \text{certified error bound}$$

$$\text{physical constraints} \mapsto \text{loss terms.}$$

Fibre:

$$\text{observed data} \mapsto \text{insieme of the PDE compatibili.}$$

9.7 E7. Network inverse problems–tomography–machine learning–invariants

Nodes:

$$\text{signals}$$

$$\text{immagini}$$

$$\text{varieties}$$

$$\text{graphs}$$

$$\text{groups}$$

$$\text{Radon transforms}$$

$$\text{scattering transforms}$$

$$\text{graph neural networks}$$

$$\text{kernel methods}$$

$$\text{statistical queries}$$

$$\text{invariant theory.}$$

Edges:

$$X \mapsto \mathcal{O}_{\leq r}(X)$$

$$G \curvearrowright X \mapsto k[X]^G_{\leq d}$$

$$\text{graph} \mapsto \text{message-passing features}$$

$$\text{signal} \mapsto \text{wavelet/scattering coefficients}$$

$$\text{measure} \mapsto \text{statistical query oracle.}$$

Question:

$$\text{what can a bounded class of observables see?}$$

9.8 E8. Network universal inverse problems

Nodes:

$$\text{object nascosto}$$

$$\text{forward map}$$

$$\text{observed data}$$

$$\text{regularization}$$

$$\text{stability estimates}$$

$$\text{Bayesian inverse problems}$$

$$\text{optimal experimental design}$$

$$\text{compressed sensing}$$

$$\text{tomography}$$

$$\text{learning-based inversion.}$$

Arco fondamentale:

$$x \mapsto F(x).$$

Examples:

$$V \mapsto \text{scattering data}$$

$$\text{domain} \mapsto \text{boundary measurements}$$

$$\text{immagine} \mapsto \text{Radon transform}$$

$$\text{metric} \mapsto \text{lens data}$$

$$\text{coefficiente PDE} \mapsto \text{Dirichlet-to-Neumann map.}$$

Principle:

$$\text{how much does it cost to distinguish two hidden objects through accessible data?}$$

9.9 E9. Network mathematical economics–equilibria–convexity–dynamics

Nodes:

fixed point theory

game theory

Nash equilibria

market equilibria

mechanism design

optimal transport economics

mean field games

variational inequalities

convex analysis

algorithmic game theory.

Edges:

game $\mapsto$ best response correspondence

best response $\mapsto$ fixed point problem

market $\mapsto$ convex program

many-agent game $\mapsto$ mean field game PDE

mechanism $\mapsto$ incentive constraints.

9.10 E10. Network mathematical biology–reaction networks–dynamics–topology

Nodes:

chemical reaction networks

ODE systems

stochastic reaction networks

mass-action kinetics

Petri nets

persistence

bifurcation theory

network motifs

phylogenetics

algebraic statistics.

Edges:

$$\text{reaction network} \mapsto \text{ODE}$$

$$\text{reaction network} \mapsto \text{Petri net}$$

$$\text{ODE} \mapsto \text{steady-state ideal}$$

$$\text{stochastic network} \mapsto \text{master equation}$$

$$\text{phylogenetic tree} \mapsto \text{statistical moof the variety.}$$

10 F. Arithmetic, Automata, Languages, Discrete Mathematics, and Computability

10.1 F1. Network computability–tilings–subshift–groups–cellular automata

Nodes:

macchine of Turing

tag systems

Wang tilings

subshifts of finite type

cellular automata

semifinitely presented groups

finitely presented groups

rewriting systems

moof the checking

reachability.

Edges:

$$\text{Turing machine} \mapsto \text{tiling system}$$

$$\text{tiling} \mapsto \text{subshift}$$

$$\text{subshift} \mapsto \text{cellular automaton}$$

$$\text{rewriting} \mapsto \text{semigroup presentation}$$

$$\text{semigroup} \mapsto \text{group embedding.}$$

Effetto:

$$\text{undecidability} \to \text{noncomputable quantitative profiles.}$$

10.2 F2. Network arithmetic–Galois–monodromy–automata–languages

Nodes:

campi of numeri

arithmetic varieties

representations of Galois

ℓ-adic cohomology

equazioni differentials

monodromy

dessins of enfants

ricorrenze lineari

sequences automatiche

formal languages

compression.

Edges:

$$X/\mathbb{Q} \mapsto \rho_{X,\ell}$$

$$X \mapsto \#X(\mathbb{F}_p)$$

covering $\mapsto$ monodromy permutation group

linear recurrence $\mapsto$ companion matrix

$\alpha \mapsto$ digit $\text{word}_b(\alpha)$

sequence $\mapsto$ automaton/minimal transducer.

Questions:

how much arithmetic is visible from a digit prefix?

how much geometry is visible from counts modulo small primes?

10.3 F3. Network automorphic forms–L-functions–spectral PDE–arithmetic

Nodes:

modular forms

automorphic forms

representations automorfe

L-functions

trace formulas

Shimura varieties

Galois representations

spectral theory on locally symmetric spaces

periods.

Edges:

$$f \mapsto L(f, s)$$

$$f \mapsto (a_n(f))_{n \leq N}$$

$$\pi \mapsto \rho_{\pi,\ell}$$

$$\Gamma \backslash G/K \mapsto \mathrm{Spec}(\Delta)$$

$$\text{trace formula} \mapsto \text{comparison of spectra.}$$

10.4 F4. Network formal languages–automata–spectral analysis–symbolic dynamics

Nodes:

regular languages

context-free languages

weighted automata

formal power series

$$\text{subshifts}$$

$$\text{transfer matrices}$$

$$\text{thermodynamic formalism}$$

$$\text{zeta functions}$$

$$\text{compression}$$

$$\text{random walks on automata.}$$

Edges:

$$\mathcal{A} \mapsto L(\mathcal{A})$$

$$\mathcal{A} \mapsto A_{\mathcal{A}}$$

$$A_{\mathcal{A}} \mapsto \mathrm{Spec}(A_{\mathcal{A}})$$

$$L \mapsto \sum_n a_n z^n$$

$$\text{subshift} \mapsto \zeta(z).$$

Scheme:

$$\text{languages} \to \text{matrices} \to \text{spectra} \to \text{dynamics} \\ \to \text{analysis.}$$

10.5 F5. Network finite observable profiles: quotients, truncations, reductions, completions

Nodes:

$$\text{objects infiniti}$$

$$\text{finite quotients}$$

$$\text{jets}$$

$$\text{truncations}$$

$$\text{reductions mod } p$$

$$\text{completions profiniti}$$

$$p\text{-adic completions}$$

$$\text{localizations}$$

$$\text{finite-dimensional approximations.}$$

Edges:

$$X \mapsto X_{\leq r}$$

$$G \mapsto G/N$$

$$X/\mathbb{Z} \mapsto X_{\mathbb{F}_p}$$

$$R \mapsto R/\mathfrak{m}^k$$

$$A \mapsto A_n$$

$$X \mapsto \widehat{X}.$$

Idea:

every theory has finite observable profiles;
complexity is the profile by which they reveal the object.

10.6 F6. Network codes–lattices–arithmetic–mathematical cryptography

Nodes:

$$\text{lattices}$$

$$\text{linear codes}$$

$$\text{codes algebro-geometrici}$$

$$\text{geometry of numbers}$$

$$\text{isogeny graphs}$$

$$\text{modular forms}$$

$$\text{finite fields}$$

$$\text{group actions}$$

hard homogeneous spaces

proof systems.

Edges:

$$C \mapsto \Lambda_C$$

$$X/\mathbb{F}_q \mapsto C_X$$

isogeny graph $\mapsto$ expander/random walk

lattice $\mapsto$ theta series

constraint system $\mapsto$ interactive proof object.

Observables:

$$[n, k, d]$$

theta series

spectrum

weight enumerator.

10.7 F7. Network symmetry–species–generating functions–representations

Nodes:

combinatorial species

operadi

symmetric functions

S_n-representations

Hilbert schemes

moduli spaces

generating functions

D-finite functions

recurrences

topological recursion.

Edges:

$$\mathcal{S} \mapsto F_{\mathcal{S}}(x)$$

$$\mathcal{S} \mapsto Z_{\mathcal{S}}$$

$$X_n \mapsto H^*(X_n)$$

sequence $\mapsto$ linear recurrence/differential equation

moduli problem $\mapsto$ generating series.

Question:

how much of a structure is visible from its generating function?

10.8 F8. Network geometric group theory–analysis–spectra–probability

Nodes:

groups finitamente generati

Cayley graphs

random walks

growth functions

isoperimetric profiles

property T

coarse embeddings

boundaries

operator algebras

L^2-cohomology.

Edges:

$$(G, S) \mapsto \mathrm{Cay}(G, S)$$

$$G \mapsto \gamma_G(n)$$

$$G \mapsto \text{return probabilities}$$

$$G \mapsto C_r^*(G)$$

$$G \mapsto \beta_i^{(2)}(G).$$

10.9 F9. Network discrete-continuous: graphs, meshes, manifolds, operators

Nodes:

graphs

simplicial complexes

meshes

manifolds

discrete Laplacians

finite element operators

graph neural operators

spectral convergence

Gromov–Hausdorff convergence.

Edges:

$$M \mapsto \mathcal{T}_h(M)$$

$$\mathcal{T}_h \mapsto L_h$$

$$L_h \mapsto \mathrm{Spec}(L_h)$$

$$G_n \mapsto M$$

$$\text{graph signal} \mapsto \text{continuum function}.$$

10.10 F10. Network graph limits–continuous analysis–extremal combinatorics

Nodes:

finite graphs

graphons

hypergraphons

flag algebras

cut norm

exchangeable arrays

dense graph limits

sparse graph limits

property testing.

Edges:

$$G_n \mapsto W$$

$$G \mapsto (t(F,G))_F$$

$$W \mapsto T_W$$

$$W \mapsto \mathrm{Spec}(T_W)$$

exchangeable array $\mapsto$ graphon representation.

11 G. Quantum and Noncommutative Networks

11.1 G1. Network quantum–noncommutative–operatorial–tensor networks

Nodes:

quantum states

quantum channels

local Hamiltonians

$$\text{tensor networks}$$

$$\text{stabilizer formalism}$$

$$C^*\text{-algebras}$$

$$\text{von Neumann algebras}$$

$$\text{nonlocal games}$$

$$\text{semidefinite programming}$$

$$\text{noncommutative polynomial optimization}$$

$$\text{modular tensor categories.}$$

Edges:

$$\text{nonlocal game} \mapsto \text{operator relations}$$

$$H = \sum H_i \mapsto \text{quantum CSP}$$

$$\psi \mapsto \text{tensor network ansatz}$$

$$\text{operator inequalities} \mapsto \text{SDP hierarchy}$$

$$\text{TQFT} \mapsto \text{quantum circuit representation.}$$

11.2 G2. Network quantum many-body–limit PDE–tensor networks–operators

Nodes:

$$\text{many-body quantum systems}$$

$$\text{local Hamiltonians}$$

$$\text{mean field limits}$$

$$\text{Hartree equation}$$

$$\text{Gross–Pitaevskii}$$

$$\text{density matrices}$$

$$\text{BBGKY hierarchy}$$

$$\text{tensor networks}$$

$$\text{entanglement entropy}$$

$$\text{operator algebras.}$$

Edges:

$$\Psi_N \mapsto \gamma_N^{(k)}$$

$$\gamma_N^{(k)} \mapsto \gamma^{\otimes k}$$

$$\Psi_N \mapsto \text{Hartree/NLS limit}$$

$$\Psi \mapsto \text{MPS/PEPS approximation}$$

$$H \mapsto \text{ground state} \mapsto \text{entanglement profile.}$$

Fibre:

$$\Psi \mapsto {\gamma^{(k)}}_{k \le r}$$

forgets higher correlations.

11.3 G3. Network operator algebras–index–K-theory–coarse geometry

Nodes:

$$C^*\text{-algebras}$$

$$\text{von Neumann algebras}$$

$$\text{Fredholm modules}$$

$$\text{spectral triples}$$

$$K\text{-theory}$$

$$K\text{-homology}$$

$$\text{index theory}$$

$$\text{coarse geometry}$$

$$\text{group } C^*\text{-algebras}$$

$$\text{noncommutative geometry.}$$

Edges:

$$X \mapsto C(X)$$

$$\Gamma \mapsto C^*(\Gamma)$$

$$(M, D) \mapsto \text{spectral triple}$$

$$A \mapsto K_*(A)$$

$$D \mapsto \text{ind}(D).$$

12 H. Metric, Convex, and Optimization Networks

12.1 H1. Network metric–embedding–graphs–Banach–algorithms

Nodes:

$$\text{graphs}$$

$$\text{metric finite}$$

$$\text{Banach spaces}$$

$$\text{manifolds metric}$$

$$\text{expanders}$$

$$\text{optimal transport}$$

$$\text{metric embeddings}$$

$$\text{sketching}$$

$$\text{nearest neighbor search}$$

$$\text{communication complexity.}$$

Edges:

$$G \mapsto d_G$$

$$(X, d) \mapsto \ell_p^n$$

$$(X, d) \mapsto \mathrm{Lip}_1(X)$$

$$\mu, \nu \mapsto W_p(\mu, \nu)$$

$$\text{communication problem} \mapsto \text{matrix/metric.}$$

12.2 H2. Network convexity–SOS–moments–optimization–verification data

Nodes:

semialgebraic sets

polynomial optimization

sum-of-squares

moment problems

convex bodies

linear programming

semidefinite programming

control theory

Lyapunov functions

combinatorial optimization

proof complexity.

Edges:

$$S = f_i \geq 0 \mapsto \text{quadratic module}$$

$$\text{polynomial optimization} \mapsto \text{moment relaxation}$$

$$\text{polytope} \mapsto \text{slack matrix}$$

$$\text{dynamical system} \mapsto \text{Lyapunov verification data}$$

$$\text{combinatorial problem} \mapsto \text{LP/SDP hierarchy.}$$

12.3 H3. Network convex geometry–isoperimetry–concentration–probability

Nodes:

convex bodies

Brunn–Minkowski theory

isoperimetric inequalities

log-concave measures

concentration of measure

transport inequalities

random polytopes

asymptotic geometric analysis

high-dimensional probability.

Edges:

$$K \mapsto h_K$$

$$K \mapsto \mathrm{Vol}(K + tB)$$

$$\mu \mapsto \text{concentration profile}$$

$$K \mapsto \text{isotropic constant}$$

$$\mu, \nu \mapsto T_{\mu \to \nu}.$$

12.4 H4. Network tropical algebra–amoebas–asymptotic analysis

Nodes:

complex varieties

amoebas

tropical varieties

Newton polytopes

$$\text{large deviations}$$

$$\text{WKB asymptotics}$$

$$\text{idempotent analysis}$$

$$\text{max-plus algebra}$$

$$\text{optimization}$$

$$\text{control.}$$

Edges:

$$z \mapsto \log |z|$$

$$\varepsilon \log(e^{f/\varepsilon} + e^{g/\varepsilon}) \mapsto \max(f, g)$$

$$\text{Hamilton–Jacobi} \mapsto \text{max-plus linearity}$$

$$\text{polynomial} \mapsto \text{Newton polytope.}$$

13 Composite Mega-networks

Le networks precedenti diventano potentissime quando compongono.

13.1 M1. Mega-network syntax–geometry–complexity

$$\text{finite presentation} \to \text{category of models} \to \text{representation space} \to \text{invariant quotient}$$
$$\to \text{tropicalization} \to \text{matroid/fan} \to \text{CSP} \to \text{proof system.}$$

Overall transfer:

$$\text{complexity of classificazione} \to \text{complexity of the invariants} \to \text{complexity tropical}$$
$$\to \text{complexity CSP} \to \text{complexity of prova.}$$

13.2 M2. Mega-network topology–quantum–operators–algorithms

$$\text{varieties/knots} \to \text{fundamental groups} \to \text{representations} \to \text{operator algebras}$$
$$\to \text{TQFT} \to \text{quantum circuits} \to \text{tensor networks} \to \text{SDP relaxations.}$$

Observables lungo the network:

$$\text{characters} \to \text{operator traces} \to \text{quantum amplitudes} \to \text{correlations}$$
$$\to \text{moments SDP.}$$

13.3 M3. Mega-network arithmetic–dynamics–languages–compression

$$\text{number/arithmetic variety} \to \text{finite reductions} \to \text{sequences} \to \text{automata}$$
$$\to \text{languages} \to \text{compression} \to \text{complexity descrittiva.}$$

Observables:

small primes

digital prefixes

factors of words

dimension of the automaton

parse size

entropy subword.

Question:

when does an arithmetic object produce linguistically simple observable profiles?

13.4 M4. Mega-network universal local-to-global

$$\text{object globale} \to \text{cover locale} \to \text{presheaf} \to \text{cohomology}$$
$$\to \text{obstruction class} \to \text{verification data} \to \text{algorithm/proof.}$$

Principle:

$$\boxed{\text{every global failure can be presented as a gluing failure.}}$$

13.5 M5. Mega-network deformation–verification datum–formalization

$$\text{geometric problem} \to \text{dg-Lie controller} \to \text{obstruction tower} \to \text{finite verification data}$$
$$\to \text{formal proof} \to \text{machine-checkable object.}$$

Observables:

$$H^0, H^1, H^2, \ldots$$

Verification data:

vanishing

obstruction nonzero

formal smoothness

rigidity

extension to order n.

13.6 M6. Composite analytic mega-network

Prima catena:

$$\text{functions/distributions} \to \text{Fourier/wavelet coefficients} \to \text{function spaces} \to \text{operators}$$
$$\to \text{PDE} \to \text{semigroups} \to \text{kernel} \to \text{spectra}$$
$$\to \text{geometry} \to \text{topology}.$$

Seconda catena:

$$\text{PDE singolare} \to \text{microlocal data} \to \text{simplectic geometry} \to \text{sheaf category}$$
$$\to \text{Floer/Fukaya} \to \text{mirror algebra}.$$

Terza catena:

$$\text{rumore} \to \text{random distributions} \to \text{SPDE} \to \text{regularity structures}$$
$$\to \text{Hopf algebras} \to \text{renormalization} \to \text{QFT}.$$

Quarta catena:

$$\text{microstructure} \to \text{homogenized tensor} \to \text{effective PDE} \to \text{macroscopic observable}$$
$$\to \text{inverse design}.$$

Quinta catena:

$$\text{dynamical system} \to \text{transfer operator} \to \text{spectrum} \to \text{zeta function}$$
$$\to \text{entropy/pressure} \to \text{large deviations}.$$

13.7 M7. Mega-network discrete–continuous–logical–numerical

$$\text{finite graph} \to \text{metric measure approximation} \to \text{continuum space} \to \text{PDE operator}$$
$$\to \text{discretized operator} \to \text{numerical solution} \to \text{interval verification data} \to \text{formal proof}.$$

Ciclo:

$$\text{discrete} \to \text{continuous} \to \text{discrete} \to \text{verification datum}.$$

13.8 M8. Mega-network arithmetic–analytic–spectral

$$\text{arithmetic variety} \to \text{rappresentazione of Galois} \to L\text{-function} \to \text{automorphic representation}$$
$$\to \text{spectral theory} \to \text{trace formula} \to \text{geometric cycles}.$$

Observables:

$$\text{Frobenius traces}$$
$$\text{Euler factors}$$
$$\text{Fourier coefficients}$$
$$\text{eigenvalues}$$
$$\text{periods}$$
$$\text{intersection numbers}.$$

13.9 M9. Mega-network probability–PDE–optimization–learning

$$\text{data distribution} \to \text{empirical measure} \to \text{gradient flow} \to \text{PDE mean-field}$$
$$\to \text{risk minimization} \to \text{generalization verification data} \to \text{algorithm.}$$

Edges:

$$\mu \to \mathrm{Ent}(\mu)$$

$$\mu_t \to \partial_t \mu_t = \nabla \cdot (\mu_t \nabla V)$$

$$\text{training} \to \text{mean-field PDE}$$

$$\text{PDE} \to \text{convergence rate.}$$

13.10 M10. Mega-network analytic local-to-global

$$\text{local estimates} \to \text{patching} \to \text{global regularity} \to \text{compactness}$$
$$\to \text{existence} \to \text{verification data.}$$

Nodes:

$$\text{elliptic estimates}$$

$$\text{partitions of unity}$$

$$\text{Sobolev embeddings}$$

$$\text{compactness}$$

$$\text{weak convergence}$$

$$\text{defect measures}$$

$$\text{concentration compactness.}$$

13.11 M11. Mega-network of bounded observation

Scheme universal:

$$X \mapsto \mathcal{O}_{\le r}(X).$$

Examples:

$$u \mapsto \widehat{u}|_{|\xi| \le \Lambda}$$

$$M \mapsto \lambda_1, \ldots, \lambda_N$$

$$\mu \mapsto (m_\alpha)_{|\alpha| \le d}$$

$$T \mapsto \mathrm{Tr}(T^k)_{k\le N}$$

$$f \mapsto \text{samples } f(x_i)$$

$$P \mapsto \text{boundary response up to time } T$$

$$\text{graph} \mapsto \text{subgraph counts up to size } k$$

$$\text{structure} \mapsto \text{sentences of quantifier rank } \le q.$$

Principle:

$$\boxed{\begin{array}{c}\text{every bounded observation induces a partition;}\\ \text{complexity is the geometry of those classes.}\end{array}}$$

14 Most Fertile Directions

The strongest directions that emerge are the following.

14.1 Universal sheafification of complexity problems

Dato a problema globale (P), costruire:

$$P \mapsto \mathcal{F}_P.$$

Poi misurare:

$$\text{minimo raggio locale or degree coomological che vede the fallimento.}$$

Applicabile a:

$$\text{CSP}$$

$$\text{database}$$

$$\text{distributed computing}$$

$$\text{proof complexity}$$

$$\text{matching}$$

$$\text{graph coloring}$$

$$\text{SAT}$$

$$\text{learning locale}$$

$$\text{bundle theory.}$$

14.2 Tensorization of non-tensorial structures

Cercare the tensore nascosto:

$$\text{groups} \to \text{tensore of commutatore}$$

$$\text{topological spaces} \to \text{cup-product tensor}$$

$$\text{PDE} \to \text{kernel tensor}$$

$$\text{database joins} \to \text{incidence tensor}$$

$$\text{proof systems} \to \text{coefficient tensor}$$

$$\text{quantum states} \to \text{entanglement tensor}$$

$$\text{statistical models} \to \text{probability tensor.}$$

Poi usare:

$$\text{rank}$$

$$\text{border rank}$$

$$\text{slice rank}$$

$$\text{flattening rank}$$

per trasferire lower bounds.

14.3 Fibres as primary objects

Non studiare solo:

$$I : X \to Y,$$

ma:

$$\mathrm{Fib}_I(y).$$

Examples:

$$\text{spaces with the same homology}$$

$$\text{graphs with same spectrum}$$

$$\text{modelli statistici with same moments}$$

$$\text{varieties with same tropicalization}$$

groups with same finite quotients up to a N

functions with the same low coefficients

PDEs with the same boundary data

neural networks with the same function.

14.4 Transfer of verification data

Per every arco chiedere:

can target verification data be pulled back?

can source verification data be pushed forward?

verification data to study:

SOS

SDP

energy

index

cohomology

entropy

Lyapunov

barrier

interval enclosure

formal proof term.

14.5 Budgeted Morita Theory

Two theories are equivalent with budget:

$$A \simeq_{\mathcal{H}} B.$$

Possible classes of overhead:

$$\text{linear}$$

$$\text{polynomial}$$

$$\text{quasi-polynomial}$$

$$\text{elementary}$$

$$\text{computable}$$

$$\text{non-computable.}$$

This direction makes it possible to classify not only objects, but whole theories.

14.6 Microlocal–sheaf–PDE–symplectic

$$\text{PDE singularities} \to WF \to \text{Lagrangians} \to \text{microlocal sheaves} \\ \to \text{Fukaya categories.}$$

Possible use:

$$\text{analytic lower bounds} \to \text{categorical lower bounds.}$$

14.7 SPDE–regularity structures–Hopf algebras–QFT

$$\text{noise} \to \text{singular PDE} \to \text{renormalization} \to \text{Hopf algebra} \\ \to \text{field theory.}$$

Here the correct presentation is the renormalized object, not the raw datum.

14.8 Homogenization–inverse design–optimization–learning

$$\text{microstructure} \to \text{effective tensor} \to \text{target material} \to \text{inverse problem} \\ \to \text{generative design.}$$

The fibre contains all equivalent microscopic materials.

14.9 Computable analysis–proof mining–verified numerics

$$\text{existence theorem} \to \text{effective modulus} \to \text{numerical scheme} \to \text{interval verification data} \\ \to \text{formal proof.}$$

This is the network that turns qualitative analysis into verifiable mathematics.

14.10 Dynamical systems–transfer operators–zeta–spectral geometry

$$T \to \mathcal{L}_T \to \mathrm{Spec} \to \zeta_T$$
$$\to \text{periodic orbit data.}$$

It is an ideal network for transporting dynamics into spectral analysis.

14.11 Banach geometry–metric embeddings–algorithms

$$\text{metric spaces} \to \text{Banach embeddings} \to \text{distortion} \to \text{sketching}$$
$$\to \text{communication complexity.}$$

It is a natural network for algorithmic lower bounds.

14.12 Riemann–Hilbert–D-modules–monodromy–asymptotics

$$\text{PDE/ODE} \to D\text{-module} \to \text{perverse sheaf} \to \text{monodromy}$$
$$\to \text{Stokes data.}$$

This turns differential analysis into constructible topology and algebra.

15 Operational Sheet for Building a Transfer Package

For each edge of the atlas one should fill in a sheet.

15.1 Standard Sheet

Dato:

$$\mathfrak{T} : A \to B,$$

specificare:

Source Objects

$$\mathcal{X}_A.$$

Target Objects

$$\mathcal{X}_B.$$

Source Descriptions

$$\mathcal{D}_A.$$

Target Descriptions

$$\mathcal{D}_B.$$

Compiler

$$\widehat{F} : \mathcal{D}_A \to \mathcal{D}_B.$$

Presentational Overhead

$$\kappa_B(\widehat{F}(d)) \le f(\kappa_A(d)).$$

Target Observables

$$\mathfrak{O}_B.$$

Observational Pullback

$$F^* : \mathfrak{O}_B \to \mathfrak{O}_A.$$

Observational Overhead

$$\chi_A(F^*O) \leq g(\chi_B(O)).$$

Verification Data

$$V_A \leftrightarrow V_B.$$

Fibres

$$F^{-1}(y).$$

Moves in the Fibre

generators, moves, deformations, homotopies, refinements.

Bridge profile

$$B_y(n) = \text{maximum/minimum cost for navigating in the fibre with budget } n.$$

Measures

$$\mu_A \mapsto \mu_B.$$

Limits

$$X_n \to X \quad \mapsto \quad F(X_n) \to F(X).$$

Fertile Questions

separazione

indistinguishability

normalizzazione

noncomputability

lower bounds

moduli

Verification Data

fibre

distortion.

16 Final Global Map

The previous linear picture is useful as a mnemonic, but it is too small to represent the atlas: it hides algebra, representation theory, graph and matroid methods, moduli, arithmetic geometry, and several discrete-continuous bridges. A better final map should be read as three complementary layers.

First, the large thematic regions are:

$$\begin{gathered}\text{logic, syntax, proof systems, formalization}\\ \to \text{algebra, representations, moduli, homological algebra}\\ \to \text{combinatorics, graphs, matroids, codes, discrete structures}\\ \to \text{geometry, topology, sheaves, persistence, categories}\\ \to \text{analysis, PDE, spectra, dynamics, probability}\\ \to \text{arithmetic, automata, languages, symbolic dynamics, computability}\\ \to \text{quantum, operator algebras, optimization, learning, numerics.}\end{gathered}$$

Second, the main access hubs are:

syntax	presentations	normal forms	compilers
observables	moments	rank profiles	spectra
rings/algebras	modules	representations	moduli
graphs	matroids	codes	finite structures
sheaves	cohomology	obstructions	local-to-global data
operators	kernels	semigroups	resolvents
probability	entropy	large deviations	optimal transport
numerics	approximations	verification data	formal proofs.

Third, the atlas is organized by bridges rather than by isolated subjects:

$$\begin{gathered}\text{syntax} \leftrightarrow \text{semantics} \leftrightarrow \text{CSP/database/proof complexity},\\ \text{algebra} \leftrightarrow \text{representations} \leftrightarrow \text{moduli} \leftrightarrow \text{invariants},\\ \text{combinatorics} \leftrightarrow \text{graphs} \leftrightarrow \text{matroids} \leftrightarrow \text{extremal/counting methods},\\ \text{linear algebra} \leftrightarrow \text{matroids} \leftrightarrow \text{codes} \leftrightarrow \text{rank methods},\\ \text{graphs} \leftrightarrow \text{limits} \leftrightarrow \text{operators} \leftrightarrow \text{spectra},\\ \text{geometry} \leftrightarrow \text{sheaves} \leftrightarrow \text{cohomology} \leftrightarrow \text{obstructions},\\ \text{PDE} \leftrightarrow \text{semigroups} \leftrightarrow \text{kernels} \leftrightarrow \text{probability},\\ \text{arithmetic} \leftrightarrow \text{Galois/automata/languages} \leftrightarrow \text{compression},\\ \text{optimization} \leftrightarrow \text{convexity/SOS/moments} \leftrightarrow \text{verification data.}\end{gathered}$$

This map is still not a theorem. It is a navigation chart: every arrow represents a family of possible transfer packages, and every useful package must specify the compiler, observable pullback, verification data, fibre geometry, and overhead functions involved.

17 Final Conceptual Summary

The atlas can be summarized in seven layers.

17.1 Layer 1: objects

functions, spaces, groups, varieties, operators, measures, categories,
graphs, models, processes, dynamical systems, PDEs,
logical theories, neural networks, quantum objects.

17.2 Layer 2: presentations

equations, generators and relations, coefficients, kernels,
triangulations, local data, graphs, matrices, tensors,
programs, networks, schemes, samples.

17.3 Layer 3: observables

spectra, moments, entropies, cohomology, homology,
samples, queries, correlations, invariants, heat kernels,
traces, barcodes, ranks, dimensions, growth profiles.

17.4 Layer 4: verification data

SOS and SDP data, energy bounds, index data, convex dual data,
cohomological obstructions, a priori estimates, stability data,
entropy data, Lyapunov data, barriers, interval enclosures,
proof terms and formal proof data.

17.5 Layer 5: fibres

objects indistinguishable under bounded observables,
equivalent descriptions and non-identifiable parameters,
microstructures with the same limit,
functions with the same samples,
operators with the same truncated spectrum,
graphs with the same small subgraphs,
theories with the same finite models,
neural networks with the same function,
PDEs with the same boundary data.

17.6 Layer 6: scales

cutoff, precision, time, frequency, mesh, quantifier rank,
dimension, degree, number of samples, number of states,
SDP level, perturbative order, semiclassical scale,
microscopic and macroscopic scales.

17.7 Layer 7: transfer

compilers between descriptions, pullbacks of observables,
pushforwards and pullbacks of verification data,
transfer of fibres, limits, measures, regularity, scales,
transfer of lower bounds.

18 Final Formula of the Atlas

The whole project can be condensed as follows:

$$\boxed{\text{every mathematical theory has many observable profiles;}}$$

$$\boxed{\text{every observable profile is produced by a system of observables;}}$$

$$\boxed{\text{every system of observables induces fibres;}}$$

$$\boxed{\text{every fibre contains hidden information;}}$$

$$\boxed{\begin{array}{c}\text{every transfer package measures the cost of passing between}\\ \text{observable profiles, descriptions, verification data, and fibres;}\end{array}}$$

$$\boxed{\text{complexity is the quantitative geometry of access to objects.}}$$

In an even shorter form:

$$\boxed{\begin{array}{c}\text{Presentation Theory} + \text{Transfer Packages}\\ \Longleftrightarrow\\ \text{global geometry of mathematical observable profiles}\\ \text{and of their translation costs.}\end{array}}$$

This atlas collects a network of networks connecting logic, algebra, combinatorics, geometry, analysis, probability, topology, arithmetic, algorithms, mathematical physics, optimization, learning, numerics, and formalization through a single language: **descriptions, observables, verification data, fibres, scales, and controlled transfers**.

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